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Independent sets in direct products of vertex-transitive graphs

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ABSTRACT

The direct product $G \times H$ of graphs G and H is defined by

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{[(u_1, v_1), (u_2, v_2)]: (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\}.$$

In this paper, we will prove that

$$\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$$

holds for all vertex-transitive graphs G and H, which provides an affirmative answer to a problem posed by Tardif (1998) [11]. Furthermore, the structure of all maximum independent sets of $G \times H$ is determined.

1. Introduction

Let G and H be two graphs. The direct product $G \times H$ of G and H is defined by

$$V(G \times H) = V(G) \times V(H)$$

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and

$$E(G \times H) = \{[(u_1, v_1), (u_2, v_2)]:\ (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\}.$$

It is easy to see this product is commutative and associative, and the product of more than two graphs is well defined. For a graph G, the products $G^n = G \times G \times \cdots \times G$ is called the n-th power of G.

An interesting problem is the independence number of $G \times H$. It is clear that if I is an independent set of G or H, then the preimage of I under projections is an independent set of $G \times H$, and so $\alpha(G \times H) \geqslant \max\{\alpha(G)|H|, \alpha(H)|G|\}$. Here $|G|$ denotes the order of G, i.e., $|V(G)|$. It is natural to ask whether the equality holds or not. In general, the equality does not hold for non-vertex-transitive graphs (see [7]). So Tardif [11] posed the following problem.

Problem 1.1. (See Tardif [11].) Does the equality

$$\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$$

hold for all vertex-transitive graphs G and H?

Furthermore, it immediately raises another interesting problem:

Problem 1.2. When $\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$, is every maximum independent set of $G \times H$ the preimage of an independent set of one factor under projections?

If the answer to Problem 1.2 is yes, we then say the direct product $G \times H$ is *MIS-normal* (maximum-independent-set-normal). Furthermore, the direct product $G_1 \times G_2 \times \cdots \times G_n$ is said to be MIS-normal if every maximum independent set of it is the preimage of an independent set of one factor under projections.

The two problems have received some attention. Frankl [6] and Valencia-Pabon and Vera [12] solved Problem 1.1 for Kneser graphs and circular graphs, respectively. Ahlswede et al. [1] generalized Frankl's results. Ku and Wong [9] investigated the structure of maximum independent sets in direct products of permutation graphs; Wang and Yu [13] proved that both Problems 1.1 and 1.2 have positive answers if one of G and H is a bipartite graph. Larose and Tardif [10] investigated the structures of maximum independent sets in powers of circular graphs, Kneser graphs and truncated simplices. For an arbitrary vertex-transitive graph G, they asked whether or not G^n is MIS-normal for all $n \geqslant 2$ if G^2 is MIS-normal. This question has been answered positively independently by Ku and McMillan [8] and the author [15].

Given a graph G and a real number r, a *fractional r-coloring* of G is a mapping f which assigns to each independent set I of G a real number $f(I) \in [0, 1]$ so that $\sum f(I) = r$ and for any vertex v, $\sum_{v \in I} f(I) \geqslant 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the minimum r such that G has a fractional r-coloring. It is well known that if G is a vertex transitive graph, then $\chi_f(G) = |V(G)|/\alpha(G)$. A generalization of Problem 1.1 is studied in [16], where the following question is asked: Is it true that for any graphs G and H, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$? After the original version of this paper, this question was answered positively in [17], which implies a positive solution to Problem 1.1.

In this paper we shall solve both Problem 1.1 and Problem 1.2. To state our results we need to introduce some notations and notions.

For a graph G, let $I(G)$ denote the set of all maximum independent sets of G. Given a subset A of $V(G)$, we define

$$N_G(A) = \{b \in V(G):\ (a, b) \in E(G) \text{ for some } a \in A\},$$

$$N_G[A] = N_G(A) \cup A \quad \text{and} \quad \overline{N}_G[A] = V(G) - N_G[A].$$

If G is clear from the context, for simplicity, we will omit the index G.

In [15], by the so-called "No-Homomorphism" lemma of Albertson and Collins [2] we proved the following result.

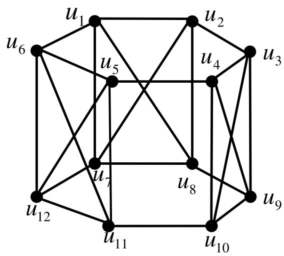

Fig. 1. Graph G.

Proposition 1.3. *Let G be a vertex-transitive graph. Then, for every independent set A of G, $\frac{|A|}{|N_G[A]|} \leqslant \frac{\alpha(G)}{|V(G)|}$. Equality implies that $|S \cap N_G[A]| = |A|$ for every $S \in I(G)$, and in particular $A \subseteq S$ for some $S \in I(G)$.*

An independent set A in G is said to be *imprimitive* if $|A| < \alpha(G)$ and $\frac{|A|}{|N[A]|} = \frac{\alpha(G)}{|V(G)|}$. And G is called *IS-imprimitive* if G has an imprimitive independent set. In any other cases, G is called *IS-primitive*.

Note that a disconnected vertex-transitive graph G is IS-imprimitive. Hence an IS-primitive vertex-transitive graph G must be connected. But, conversely, an IS-imprimitive graph is not necessarily disconnected. For example, the graph G in Fig. 1 is connected and $\{u_1, u_{12}\}$ is an imprimitive independent set of it.

The following theorem is the main result of this paper.

Theorem 1.4. *Let G and H be two vertex-transitive graphs with $\frac{\alpha(G)}{|G|} \geqslant \frac{\alpha(H)}{|H|}$. Then*

$$\alpha(G \times H) = \alpha(G)|H|,$$

and exactly one of the following holds:

(i) *$G \times H$ is MIS-normal,*
(ii) *$\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and one of G or H is IS-imprimitive,*
(iii) *$\frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|}$ and H is disconnected.*

Note that when condition (ii) holds, $G \times H$ is not MIS-normal. In fact, if $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and A is an imprimitive independent set of G, then for every $I \in I(H)$, it is easy to see that $S = (A \times V(H)) \cup (\overline{N}[A] \times I)$ is an independent set of $G \times H$ with size $\alpha(G)|H|$. While when (iii) holds, $G \times H$ is clearly not MIS-normal.

The above theorem has an immediate consequence as follows.

Corollary 1.5. *If both G and H are vertex transitive graphs, then $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.*

We leave the proof of Theorem 1.4 to the next section, while in Section 3, we discuss the MIS-normality of the direct products of more than two vertex-transitive graphs.

2. Proof of Theorem 1.4

Let S be a maximum independent set of $G \times H$. Then $|S| \geqslant \alpha(G)|H| \geqslant |G|\alpha(H)$. We will prove $|S| \leqslant \alpha(G)|H|$.

For every $a \in G$, define $X_a = \{x \in H\colon\ (a, x) \in S\}$. Then $|S| = \sum_{a \in V(G)} |X_a|$. Since S is an independent set of $G \times H$, for each $x \in X_a$ and $y \in X_b$, $(x, y) \notin E(H)$ whenever $(a, b) \in E(G)$. In this case, we say that X_a and X_b are cross-independent. This concept is equivalent to cross-intersecting families in extremal set theory. We refer to [14] for details.

In the language of cross-intersecting families, Borg et al. [3–5] introduced a decomposition of X_a as follows:

$$X_a^* = \{x \in X_a\colon\ N_H(x) \cap X_a = \emptyset\},$$
$$X_a' = \{x \in X_a\colon\ N_H(x) \cap X_a \neq \emptyset\}$$

and

$$X' = \bigcup_{a \in V(G)} X_a'.$$

Clearly, X_a^* is an independent set of H for every $a \in V(G)$. Here, the empty set is regarded as an independent set. We list all distinct X_a^*'s as $Y_1, Y_2, \dots, Y_k$, and define

$$B_i = \{a \in V(G)\colon\ X_a^* = Y_i\}, \quad i = 1, 2, \dots, k.$$

We then obtain a partition of $V(G)$ as $V(G) = B_1 \cup B_2 \cup \dots \cup B_k$. Then

$$|S| = \sum_{a \in V(G)} |X_a| = \sum_{a \in V(G)} \left(|X_a^*| + |X_a'|\right) = \sum_{i=1}^{k} \sum_{a \in B_i} |X_a^*| + \sum_{a \in V(G)} |X_a'|.$$

For $x \in V(H)$, set $A_x = \{a \in V(G)\colon\ x \in X_a'\}$. Clearly, $A_x = \emptyset$ if $x \in V(H) \setminus X'$, which implies that

$$|S| = \sum_{i=1}^{k} |Y_i||B_i| + \sum_{x \in X'} |A_x|. \tag{1}$$

For every pair $a, b \in V(G)$, it is easy to verify that $(a, b) \notin E(G)$ if $X_a' \cap X_b' \neq \emptyset$. Therefore, A_x is an independent set of G. By Proposition 1.3 we have that for each $x \in X'$,

$$|A_x| \leqslant \frac{\alpha(G)}{|V(G)|} \left|N_G[A_x]\right|, \tag{2}$$

and equality holds if and only if $|A_x| = 0$, or $|A_x| = \alpha(G)$, or A_x is an imprimitive independent set of G. Furthermore, if $x \in V(H) \setminus X'$, then $A_x = \emptyset$. Therefore, (2) holds for all $x \in V(H)$.

We claim that, for each $1 \leqslant i \leqslant k$, if $x \in N_H[Y_i]$, then $B_i \subseteq \overline{N}_G[A_x]$. In fact, suppose $x \in N_H[Y_i] = N_H(Y_i) \cup Y_i$. If $x \in N_H(Y_i)$, then there exists $y \in Y_i$ such that $(x, y) \in E(H)$ and $\{(a, x), (b, y)\} \subset S$ for every $a \in A_x$ and $b \in B_i$, hence $(a, b) \notin E(G)$ since S is an independent set; if $x \in Y_i$, then for each $a \in A_x$ and $b \in B_i$, there is a $z \in X_a$ with $(x, z) \in E(H)$ and $\{(a, z), (b, x)\} \subset S$, yielding $(a, b) \notin E(G)$. For each $b \in B_i$, $X_b' \cap N_H[Y_i] = \emptyset$ since $X_b^* = Y_i$. So $B_i \cap A_x = \emptyset$ for each $x \in N_H[Y_i]$. Thus $B_i \subseteq \overline{N}_G[A_x]$, proving the claim. From this it follows that for every $x \in X'$,

$$\sum_{i\colon\ x \in N_H[Y_i]} |B_i| \leqslant \left|\overline{N}_G[A_x]\right| = |G| - \left|N_G[A_x]\right|,$$

i.e.,

$$\left|N_G[A_x]\right| \leqslant |G| - \sum_{i\colon\ x \in N_H[Y_i]} |B_i| = \sum_{i\colon\ x \in \overline{N}_H[Y_i]} |B_i|. \tag{3}$$

Combining (2) and (3), we obtain that

$$\begin{aligned} \sum_{x \in X'} |A_x| &\leqslant \frac{\alpha(G)}{|G|} \sum_{x \in X'} \sum_{i\colon\ x \in \overline{N}_H[Y_i]} |B_i| \quad \big(\text{by (2) and (3)}\big) \\ &\leqslant \frac{\alpha(G)}{|G|} \sum_{x \in V(H)} \sum_{i\colon\ x \in \overline{N}_H[Y_i]} |B_i| \qquad (4) \end{aligned}$$

$$= \frac{\alpha(G)}{|G|} \sum_{i=1}^{k} \sum_{x \in \overline{N}_H[Y_i]} |B_i|$$

$$= \frac{\alpha(G)}{|G|} \sum_{i=1}^{k} |B_i| \left|\overline{N}_H[Y_i]\right|. \tag{5}$$

For each Y_i, by Proposition 1.3, we have

$$|Y_i| - \frac{\alpha(G)}{|G|} \left|N_H[Y_i]\right| \leqslant |Y_i| - \frac{\alpha(H)}{|H|} \left|N_H[Y_i]\right| \leqslant 0. \tag{6}$$

Combining (1), (5) and (6), we get

$$\begin{aligned}
|S| &= \sum_{i=1}^{k} |Y_i||B_i| + \sum_{x \in X'} |A_x| \\
&\leqslant \sum_{i=1}^{k} |Y_i||B_i| + \frac{\alpha(G)}{|G|} \sum_{i=1}^{k} |B_i| \left|\overline{N}_H[Y_i]\right| \\
&= \sum_{i=1}^{k} |B_i| \left(\frac{\alpha(G)}{|G|} |H| + |Y_i| - \frac{\alpha(G)}{|G|} \left|N_H[Y_i]\right| \right) \\
&= \alpha(G)|H| + \sum_{i=1}^{k} |B_i| \left(|Y_i| - \frac{\alpha(G)}{|G|} \left|N_H[Y_i]\right| \right) \\
&\leqslant \alpha(G)|H|.
\end{aligned}$$

This proves the equality in Theorem 1.4.

The maximality of $|S|$ implies that $|S| = \alpha(G)|H|$, from which it follows that (2), (3), (4) and (6) hold with equality. Also, from Proposition 1.3, (6) holds with equality means that for each $i = 1, 2, \ldots, k$ either $Y_i = \emptyset$, or $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and Y_i is either imprimitive or a maximum independent set of H. Moreover, (4) holds with equality means

$$\frac{\alpha(G)}{|G|} \sum_{x \in V(H) \setminus X'} \sum_{i:\, x \in \overline{N}_H[Y_i]} |B_i| = 0.$$

However, since $|B_i| > 0$ for every i, the above equality holds only if there are no x and i with $x \in (V(H) \setminus X') \cap \overline{N}_H[Y_i]$. In other words, $(\bigcup_{1 \leqslant i \leqslant k} \overline{N}_H[Y_i]) \subseteq X'$. On the other hand, it is clear that $X' \subseteq (\bigcup_{1 \leqslant i \leqslant k} \overline{N}_H[Y_i])$. Therefore, (4) holds with equality means

$$X' = \bigcup_{1 \leqslant i \leqslant k} \overline{N}_H[Y_i]. \tag{7}$$

We now prove that either S is the preimage under projections of a maximum independent set of G or H, or (ii) or (iii) holds. There are two cases to be considered.

Case 1: $\frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|}$. Then, equality (6) means that $Y_i = \emptyset$ for all i, and so $X' = V(H)$ by equality (7). Hence, from equality in (2) and (3) it follows that A_x is a maximum independent set of G for all $x \in V(H)$. With this assumption we have that for any $x, y \in V(H)$ with $(x, y) \in E(H)$, if $A_x \neq A_y$, there must exist $a \in A_x$ and $b \in A_y$ with $(a, b) \in E(G)$ since both A_x and A_y are maximum independent set, so $[(a, x), (b, y)] \in E(G \times H)$, contradicting $\{(a, x), (b, y)\} \subset S$. Therefore, $A_x = A_y$ whenever $(x, y) \in E(H)$, which implies that S is the preimage of a maximum independent set of G under projections if H is connected.

Case 2: $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$. Then, equality (6) means that for each index i either $|Y_i| = 0$ or $\alpha(H)$, or Y_i is an imprimitive independent set of H. If Y_i is an imprimitive independent set of H for some i, then H is IS-imprimitive. If $|Y_i| = \alpha(H)$ for all i, then $X_a = X_a^*$ is a maximum independent set of H for all $a \in V(G)$, and we can prove in a similar way to Case 1 that S is the preimage of a maximum independent set of H under projections if G is connected. We now suppose that $|Y_i| = 0$ for some i. With this assumption, then equality (7) implies $X' = V(H)$, and then equality (2) means that either A_x is either imprimitive or a maximum independent set of G for all $x \in V(H)$. If the former holds for some $x \in V(H)$, we have that G is IS-imprimitive; otherwise, the latter holds for all $x \in V(H)$, and then we can prove in the same way as in Case 1 that S is the preimage of a maximum independent set of G under projections if H is connected.

3. Concluding remark

In this section we discuss the MIS-normality of direct product of more than two graphs.

Let $n \geqslant 3$ be an integer, let $G_1, G_2, \dots, G_n$ be n connected vertex-transitive graphs with $\frac{1}{2} \geqslant \frac{\alpha(G_1)}{|G_1|} = \cdots = \frac{\alpha(G_\ell)}{|G_\ell|} > \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \geqslant \cdots \geqslant \frac{\alpha(G_n)}{|G_n|}$, and set $G = G_1 \times G_2 \times \cdots \times G_n$. From Theorem 1.4 it follows immediately that

$$\alpha(G) = \frac{\alpha(G_1)}{|G_1|}|G|.$$

By definition we see that for $i \geqslant 3$, $G_1 \times G_2 \times \cdots \times G_i$ is MIS-normal if and only if $G_1 \times \cdots \times G_{i-1}$ and $(G_1 \times \cdots \times G_{i-1}) \times G_i$ are both so. Observe that $(G_1 \times \cdots \times G_{i-1}) \times G_i$ is MIS-normal does not mean $G_1 \times G_2 \times \cdots \times G_i$ is. For example, let G be the graph of Fig. 1. It is easy to see that $(G \times G) \times K_5$ is MIS-normal and $G \times G \times K_5$ is not. From Theorem 1.4 it follows that $(G_1 \times \cdots \times G_{i-1}) \times G_i$ is MIS-normal if $i > \ell$. Repeating this process we have that $G_1 \times G_2 \times \cdots \times G_n$ is MIS-normal if $\ell = 1$. And, when $\ell \geqslant 2$ we need only discuss the MIS-normality of $G_1 \times G_2 \times \cdots \times G_\ell$. In this case we see that if one of the factors is IS-imprimitive, then the direct product is not MIS-normal. Conversely, if every G_i is assumed to be IS-primitive, the MIS-normality of $G_1 \times G_2 \times \cdots \times G_\ell$ depends on the IS-primitivity of $G_1 \times G_2 \times \cdots \times G_{\ell-1}$. We have the following two lemmas.

Lemma 3.1. *(See [15, Theorem 2.6].) Let H and K be two non-bipartite vertex-transitive graphs with $\frac{\alpha(H)}{|H|} = \frac{\alpha(K)}{|K|}$. If $H \times K$ is MIS-normal, then $H \times K$ is IS-primitive.*

A graph H is said to be *non-empty* if $E(H) \neq \emptyset$. It is well known that if H is a non-empty vertex-transitive graph, then $\frac{\alpha(H)}{|H|} \leqslant \frac{1}{2}$, and equality holds if and only if H is a bipartite graph. The following result on the IS-primitivity of bipartite graphs is given by Wang and Yu [13]. For completeness, we present a short proof here.

Lemma 3.2. *Suppose that G is a vertex-transitive bipartite graph. Then G is IS-imprimitive if and only if G is disconnected.*

Proof. It is clear that G is IS-imprimitive if G is disconnected. Conversely, if G is IS-imprimitive, then there is an imprimitive independent set A such that $\frac{|A|}{|N_G[A]|} = \frac{\alpha(G)}{|G|} = \frac{1}{2}$. Set $B = N_G(A)$. Then $|B| = |A|$ and $A \subseteq N_G(B)$. If $N_G(B) \neq A$, then we obtain that $\sum_{u \in A} d(u) \leqslant \sum_{v \in B} d(v)$, which induces a contradiction. Hence $N_G(B) = A$, that is, G is disconnected. $\square$

Applying Theorem 1.4, Lemma 3.1, the fact that the direct product of two or more bipartite graphs is disconnected, and Lemma 3.2, we obtain the following theorem.

Theorem 3.3. *Let $G_1, G_2, \dots, G_n$ be connected vertex-transitive graphs with $\frac{1}{2} \geqslant \frac{\alpha(G_1)}{|G_1|} = \cdots = \frac{\alpha(G_\ell)}{|G_\ell|} > \frac{\alpha(G_{\ell+1})}{|G_{\ell+1}|} \geqslant \cdots \geqslant \frac{\alpha(G_n)}{|G_n|}$, where $n \geqslant 2$ and $1 \leqslant \ell \leqslant n$. Then $G_1 \times G_2 \times \cdots \times G_n$ is MIS-normal if and only if one of the following holds*:

(i) $\ell = 1$;
(ii) $\ell > 1$, $\frac{\alpha(G_1)}{|G_1|} < \frac{1}{2}$ *and* $G_1, G_2, \ldots, G_\ell$ *are all IS-primitive*;
(iii) $\ell = 2$ *and* $\frac{\alpha(G_1)}{|G_1|} = \frac{1}{2}$.

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